Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Procedia - Social and Behavioral Sciences 108 (2014) 260 – 269

Procedia
Social and Behavioral Sciences

AIRO Winter 2013

Some notes on the structure of limit sets in IS-LM models

Giovanni Bella[a], Paolo Mattana[a], Beatrice Venturi[a,]*

[a]*Department of Economics and Business, University of Cagliari, Viale S. Ignazio 17, 09123 Cagliari, Italy*

Abstract

We analyze the global dynamics of the solutions of a general non-linear fixed-price disequilibrium IS-LM model, where the investment function avoids any Kaldor-type assumption. The structure of the limit sets of the model with a third order non linearity is studied. We use rigorous arguments to show that, as the bifurcation parameters vary, a wide range of dynamical behavior is displayed.

Keywords: multiple steady states; homoclinic bifurcation; oscillating solutions

1. Introduction

In order to understand the way a financial crisis can be caused by a breakdown of the dynamic stability of an economic model, according to some bifurcations mechanism, in this work we consider a Schinasi (1981, 1982) variant of a disequilibrium fixed-price IS-LM macroeconomic model as a family of a two-parameter and three-dimensional ordinary differential equations.[†] Economic nonlinear dynamic systems may exhibit an extremely rich pattern of asymptotic behavior. The simplest, and most tractable, are steady states, closed and periodic orbits (see inter al. Lorenz, 1989; Jarsulic, 1993; Benhabib and Perli, 1994; Sasakura, 1994; Mattana and Venturi, 1999; Neri and Venturi, 2007; Bella et al., 2013); but these are not the only possible outcomes. In particular, the investigation of ω-limit set, regular or chaotic, is of crucial importance to economists who care about the long run impact of

* Corresponding author. Tel.: +39-0706753333; fax: +39-070660929.
E-mail address: venturi@unica.it

[†]The IS-LM model is the baseline for any macroeconomic policy study, since it describes the equilibrium relationships between interest rates and real output, in both the market for goods and the liquidity market. In fact, whereas the market for goods is in equilibrium when investment (*I*) equals savings (*S*), the money market reaches its equilibrium when demand for money (*L*) equals the supply of money issued (*M*). The curve describing the equilibrium locus in the market for goods is therefore appealed as *IS*, while the locus for equilibrium in the money market is synthesized by the letters *LM*. The mutual equilibrium of both markets describes the couple of interest rate and real output that guarantee the general equilibrium.

doi:10.1016/j.sbspro.2013.12.836

policies and institutions. The model is crucial from a macroeconomic perspective, and the estimation of its parameters is often used to predict the future trend of the gross national outcome. Despite this, many economists have argued that the model may fail its predictions due to unwelcome situations, like as the emergence of economic fluctuations and chaotic pattern of its variables. An important economic application is given in Adachi and Nakamura (2004). They have in fact used the IS-LM model to explain the low interest rates and low prices that hit Japan for several decades. More recently, Choi and Douady (2012) provide a contribution to validate the IS-LM model and the possibility of chaotic solutions for the 2007 US financial crisis.

For example, Neri and Venturi (2007) considered a general non-linear fixed-price three-dimensional disequilibrium IS-LM model with the investment behavior as a general non-linear function avoiding any Kaldor-type assumption (see, Kaldor, 1940). They established, analytically, via the Hopf bifurcation theorem, that a stable economy could be destabilized into a stable cycle in the full $\Re^3$ dynamics (see, Wiggins, 1990, p. 276). That is to say that, the values of the adjustment parameter in the money market may affect the long run equilibrium. By using the globally analysis instrument of the Kopell-Howard Theorem (1975), Bella et al. (2013), show that an economy not satisfying the Kaldorian assumptions can also present an oscillating behavior. They analyzed a family of two parameters, two-dimensional, and pure money financing of the budget deficit IS-LM model, in which the interest rate sensitivity of savings can be made negative. In this work, they compute normal forms for the triple-zero bifurcations of a family of a two parameter systems, and determine the local bifurcations that emerge from such degeneracies.

The paper develops as follows. The second Section introduces the well-known Schinasi (1981, 1982) variant of the standard IS-LM dynamical system, and studies the long-run properties of the equilibrium. The third Section is devoted to reduce the model to canonical form, and also to discuss the bifurcations of equilibria that can be inferred from such canonical form. Appendix provides all the necessary proofs.

2. The Model

We consider a general disequilibrium, fixed price, IS-LM model with pure money financing of the budget deficit, which implies the following family of two parameters and three-dimensional first order differential equations (see also Sasakura, 1994; Neri and Venturi, 2007; Makovinyiova, 2011)

$$\begin{aligned} \dot{r} &= \delta[L(r,y)-m] \\ \dot{y} &= \alpha\left[I(y,r)-S(y^D;W)+g+T(y)\right] \\ \dot{m} &= g-T(y) \end{aligned} \tag{1}$$

where $\dot{r}=\frac{dr}{dt}$, $\dot{y}=\frac{dy}{dt}$, and $\dot{m}=\frac{dm}{dt}$. We also assume that all functions are continuously differentiable at a suitable order.

The quantities I, S, T, g, r, y and w represent (respectively), investment, savings, tax collections, government expenditure, interest rate, output (income) and wealth. Let $L(r;y)$ be the liquidity (i.e., money demand) function, depending on r, the (real) interest rate, and y the income level. As found in the related literature, we need that $L_y>0$ and $L_r<0$. For simplicity, prices are fixed at unity. Here $\alpha>0$ and

$\delta > 0$ are the adjustment parameters in the markets for liquidity and goods, respectively. Moreover, we assume that g is a positive constant.

Let $S(y^D;W)$ capture savings as function of both disposable income, y^D, and wealth, W. We assume the total amount of real money balances: $W = m$ and no bonds issued, to simplify the analysis. The tax collection function, $T(y)$, is choose to proportionally depend on income $(T = \tau y)$, where $\tau \in [0,1]$ is the proportional tax rate. Next, we define the disposable income as follows: $y^D = y - T(y)$.

We can rewrite the system in following form, without any loss of generality:

$$\begin{aligned} \dot{r} &= \delta[\gamma y - \beta r - m] \\ \dot{y} &= \alpha f(y,r,m) + \alpha(g - \tau y) \\ \dot{m} &= g - \tau y \end{aligned} \tag{2}$$

where $f(r;y;m) = I(r,y) - S(y^D;W)$.

Moreover, the liquidity function is given by $L = \gamma y - \beta r$, whereas the disposable income function is $y^D = y - T(y) = (1-\tau)y$, which are both linear in their arguments, being γ and β a measure for the sensibility of the liquidity function to changes in real income and real interest rate, respectively.

2.1. Steady states analysis

Let $P^*\left(r^*, y^*, m^*\right)$ denote the steady states of system (2), such that $\dot{r} = \dot{y} = \dot{m} = 0$.

$$\begin{aligned} \gamma y^* - \beta r^* &= m^* \\ f(y^*, r^*, m^*) &= 0 \\ g &= \tau y^* \end{aligned} \tag{3}$$

where $f(\cdot) \to \Re^+$ is a function conveniently smooth in all its arguments. We define the function as follows

$f(r,y,m) = I(r,y) - S(y^D,W).$

We also consider the case in which $\frac{df}{dy} = I_y - S_y$ changes sign in its domain, for specific values of y, and $f(\cdot)$ can have multiple intersections with the y-axis (see Makovínyiová, 2011) that is multiple steady emerge (see Fig. 1).

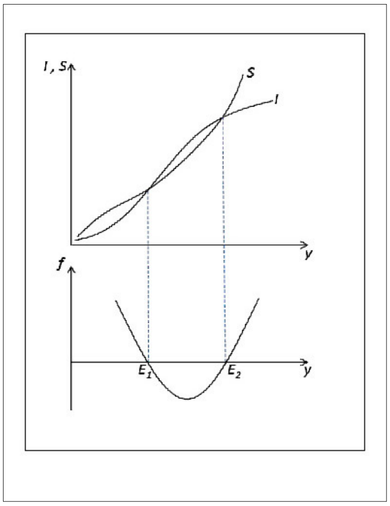

Fig. 1. Multiple equilibria

The Jacobian matrix of the system (2), evaluated at the steady state is thus given by $J(P^*) = J^*$. That is formally

$$\mathrm{J}^* = \begin{bmatrix} -\delta\beta & \delta\gamma & -\delta \\ \alpha f_r & \alpha f_y - \alpha\tau & \alpha f_m \\ 0 & -\tau & 0 \end{bmatrix} \tag{4}$$

from which we can derive

$$\begin{aligned} \mathrm{Tr}(\mathrm{J}^*) &= \alpha(f_y - \tau) - \delta\beta \\ \mathrm{Det}(\mathrm{J}^*) &= \alpha\delta\tau\left(f_r - \beta f_m\right) \\ \mathrm{B}(\mathrm{J}^*) &= \alpha\tau f_m - \alpha\delta\left[\beta\left(f_y - \tau\right) + \gamma f_r\right] \end{aligned} \tag{5}$$

that represent the trace, the determinant, and the sum of the principal minors associated to J^*, respectively.

In order to check whether system (2) may exhibit a chaotic attractor dynamics, we apply the following Shilnikov theorem:

Theorem 1 *If the third-order autonomous system (2) has two saddle-foci (of index 2) equilibrium points,* E_1 *and* E_2*, with eigenvalues associated to (4) given by* $\eta_k \in \mathrm{R}$ *and* $\sigma_k + i\omega_k \in \mathrm{C}, \quad k = 1,2$*, such that* $\sigma_1\sigma_2 > 0$ *or* $\eta_1\eta_2 > 0$*, with a further constraint* $|\eta_k| > |\sigma_k|$*, and there exists a heteroclinic orbit, connecting* E_1 *and* E_2*, then the dynamic flow in the neighborhood of heteroclinic orbit exhibits a Smale horseshoe type of chaos.*

Proof See, Wang et al. (2009).

System (2) fulfils the above theorem by solving (5) with Cardano's formula, which provides the following three roots:

$$\begin{aligned} \lambda_1 &= -\frac{\hat{a}}{3} + u + v \\ \lambda_{2,3} &= -\frac{\hat{a}}{3} - \frac{u+v}{2} \pm \sqrt{3}\,\frac{u-v}{2} i \end{aligned} \tag{6}$$

where $i = \sqrt{-1}$ is the imaginary root, $u = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}}$ and $v = \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}}$, with $p = \frac{3\hat{b} - \hat{a}^2}{3}$ and $q = \hat{c} + \frac{2\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{3}$, $\hat{a} = -Tr(J^*)$, $\hat{b} = BJ(J^*)$, and $\hat{c} = -Det(J^*)$, whereas $\Delta = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is the discriminant. For the scope of our paper, a saddle-focus (of index 2) emerges when

$$\begin{gathered} \Delta > 0 \\ \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\Delta}} < -\frac{2\hat{a}}{3} \end{gathered} \tag{7}$$

that is explicitly

$$\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6} \right)^2 > \left(\frac{\hat{a}^2 - 3\hat{b}}{9} \right)^3 \tag{8.1}$$

and

$$\sqrt[3]{-\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6} \right) + \sqrt{\Delta}} + \sqrt[3]{-\left(\frac{\hat{c}}{2} + \frac{\hat{a}^3}{27} - \frac{\hat{a}\hat{b}}{6} \right) - \sqrt{\Delta}} < -\frac{2\hat{a}}{3} \tag{8.2}$$

given

$$\begin{aligned} \hat{a} &= \delta\beta - \alpha(f_y^* - \tau) \\ \hat{b} &= \alpha\tau f_m^* - \alpha\delta\left[\beta\left(f_y^* - \tau\right) + \gamma f_r^* \right] \\ \hat{c} &= \alpha\delta\tau\left(\beta f_m^* - f_r^* \right) \end{aligned} \tag{9}$$

which guarantee the emergence of a scroll (a wing) around the equilibrium points.

3. Normal forms and bifurcations

Our first task will be to put the IS-LM in an appropriate normal form. To this scope, we use a partial unfolding of a triple-zero eigenvalue bifurcation condition to put the system in the following normal form (See appendix A)

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= w_3 \\ \dot{w}_3 &= -\hat{c}w_1 + \hat{b}w_2 - \hat{a}w_3 + s_1 w_1^2 + s_2 w_3^2 \end{aligned} \tag{10}$$

Therefore, (10) constitutes an organizing center of a codimension-three singularity where the origin exhibits a triple-zero eigenvalue, which in embryo contains several bifurcation singularity. In particular, if we expand (w_1, w_2, w_3) up to the third order degree, and annihilate all cross-product, we can re-write the system as a family of three-dimensional autonomous differential equations of the following jerk function

$$\dddot{w}_1 + \hat{a}\ddot{w}_1 - \hat{b}\dot{w}_1 + g(w_1) = 0 \tag{11}$$

whose global structure is topologically equivalent to the original system (2).

Moreover, without any loss of generality, we can also set $s_1 = \alpha\tau f_{rr}$ and $s_2 = 1$, such that system (10) may resemble the little-known nonlinear Arneodo system, a particular convenient normal form with a very rich associated dynamics, crucially depending on parameter s_1, including the emergence of a complex chaotic attractor (see, Arneodo et al., 1981).

In detail, s_1 represents the second order derivative of $f(\cdot)$ with respect to r, which can be interpreted in economics as a measure of the variation (acceleration) of speed adjustment of total inventories with respect to the real interest rate (i.e., $f_{rr} = I_{rr} - S_{rr}$), which in modulus can be greater (lower) than unity. To determine the proper sign of s_1, we need to note that, while the effect of real interest rate change on the investment function is always negative $I_{rr} < 0$, the same cannot be said for the savings functions. As suggested by standard economic theory, the impact of interest rate variations on the savings function remains still uncertain. For the purpose of our paper, we show that only when $S_{rr} < 0$, that is to say standard Kaldorian assumptions are violated, given $\alpha > 0$ and $\tau > 0$, the s_1 parameter tends to zero, and the double-scroll chaotic scenario occurs.

We are able to show these differences numerically in the following examples, including the emergence of a complex chaotic attractor where:

$$g(w_1) = -(w_1 + s_1 w_1^2 + s_2 w_3^2) \tag{12}$$

is a second order polynomial.

Referring to Makovinyiova (2011), if we consider the set of structural parameters for Slovak economy, namely $(\alpha,\beta,\gamma,\delta,\tau)\equiv(8.32,0.2,0.8,1.23,0.2)$, then $(f_r,f_y,f_m)\equiv(0.45,0.23,2.27)$, which entail also $(\varepsilon_1,\varepsilon_2,\varepsilon_3)\equiv(5,-3.8,-1)$. This allows us to show the equilibrium dynamics when s_1 is varied.

Example 1 *If* $s_1=-2$ *The hyperbolic fixed point of system (2) exhibits a single-scroll scenario in the* (w_1,w_2,w_3) *space (see Fig. 2).*

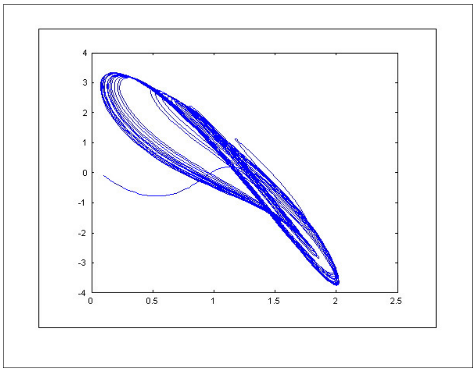

Fig. 2. The single-scroll attractor

Example 2 *If* $s_1=-0.2$ *The hyperbolic fixed point of system (2) exhibits a double scroll scenario in the* (w_1,w_2,w_3) *space (see Fig. 3).*

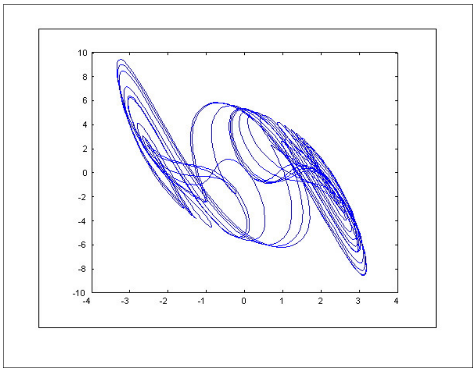

Fig. 3. The double-scroll attractor

4. Conclusions

The purpose of this paper is to understand the evolution of the financial crisis and the transition from equilibrium to chaos. The aim of the present paper is to point out some basic ideas that may be useful to prove the transition to bounded and complex behavior, and to explain how the presence of Hopf bifurcations in a general class of economic-financial models can be interesting from an economic and dynamic point of view. In this note we proved that our system satisfies the Shilnikov theorem assumptions. The nature of the growth paths in this chaotic regime were seen to depend on the initial conditions and they looked noisy, like the simple function of a stochastic process. Finally, chaos has interesting implications for the rational expectations. If the economy happens to be in the chaotic regime, then, even if economic agents know perfectly how the economy functions, they are unable to fully predict its deterministic behavior.

This finding might explain the situation where the economy exhibits a dual steady state: both with a positive growth rate of output but with different interest rates. We demonstrate that, it is possible to find a defined set of parameters for which multiple cycles emerge around each steady state, so that the economic variables start fluctuating around each long run equilibrium, until the two spirals interconnect, so that the economy is finally trapped in this chaotic oscillating pattern.

References

Adachi H., Nakamura T. (2004). A dynamic Analysis of an Economy with a zero Interest Rate Bound, Graduate school of economics, scholarly articles, vol. 1391, p.147-160.

Arneodo A., Coullet P., Tresser C. (1981). Possible New Strange Attractors With Spiral Structure. Commun. Math. Phys, 79, p. 573-579.

Bella G., Mattana P., Venturi, B. (2013). Kaldorian assumptions and endogenous fluctuations: a note on Schinasi’s IS–LM model. International Review of Economics, Vol. 60(1), p. 71-81.

Benhabib, J. and Perli, R. (1994). Uniqueness and indeterminacy: on the dynamics of endogenous growth. Journal of Economic Theory, 63, p. 113-142.

Choi Y., Douady R. (2012). Financial crisis dynamics: attempt to define a market instability indicator. Quantitative Finance, 12(9), p. 1351-1365.

Gamero E., Freire E., Rodriguez-Luis A.J., Ponce E., Algaba A. (1999). Hypernormal Form Calculation for Triple-Zero Degeneracies. Bulletin of the Belgian Mathematical Society. Simon Stevin 6, p. 357--368.

Jarsulic M. (1993). Complex dynamics in a Keynesian growth model. Metroeconomica, 44(1), p. 43-64.

Kaldor N. (1940). A Model of the Trade Cycle, The Economic Journal, 50(197), p. 78-92.

Kopell N., Howard L.N. (1975). Bifurcations and trajectories joining critical points, Advances in Mathematics 18, p. 306-358.

Lorenz H.W. (1993). Nonlinear Dynamical Economics and Chaotic Motion, 2nd ed. Springer-Verlag, New York.

Makovinyiova K. (2011). On the existence and stability of business cycles in a dynamic model of a closed economy. Nonlinear Analysis: Real World Applications 12, 1213-1222.

Mattana P., Venturi B. (1999). Existence and stability of periodic solutions in the dynamics of endogenous growth. International Review of Economics and Business, 46, 259-284.

Neri U., Venturi B. (2007). Stability and Bifurcations in IS-LM economic models. International Review of Economics, 54, p. 53-65.

Sasakura K. (1994). On the dynamic behavior of Schinasi's business cycle model, Journal of Macroeconomics 16, 423-444.

Schinasi G.J. (1981). A nonlinear dynamic model of short run fluctuations. Review of Economic Studies 48, 649-653.

Schinasi G.J. (1982). Fluctuations in a dynamic, intermediate-run IS-LM model: applications of the Poincaré-Bendixon theorem. Journal of Economic Theory 28, 369-375.

Wang J., Chen Z., Yuan Z. (2009). Existence of a new three-dimensional chaotic attractor. Chaos, Solitons and Fractals 42, p. 3053-3057.

Appendix A.

A.1. Normal form reduction

$$\begin{pmatrix} \dot{\tilde{r}} \\ \dot{\tilde{y}} \\ \dot{\tilde{m}} \end{pmatrix} = \mathbf{J}^* \begin{pmatrix} \tilde{r} \\ \tilde{y} \\ \tilde{m} \end{pmatrix} + \begin{pmatrix} \tilde{f}_1(\tilde{r}, \tilde{y}, \tilde{m}) \\ \tilde{f}_2(\tilde{r}, \tilde{y}, \tilde{m}) \\ \tilde{f}_3(\tilde{r}, \tilde{y}, \tilde{m}) \end{pmatrix} \tag{A.1}$$

$$\begin{pmatrix} \tilde{f}_1(\tilde{r}, \tilde{y}, \tilde{m}) \\ \tilde{f}_2(\tilde{r}, \tilde{y}, \tilde{m}) \\ \tilde{f}_3(\tilde{r}, \tilde{y}, \tilde{m}) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}\alpha\left[f_{rr}\tilde{r}^2 + 2 f_{ry}\tilde{r}\tilde{y} + f_{yy}\tilde{y}^2 \right] \\ 0 \end{pmatrix} \tag{A.2}$$

this allows us to make the following change of co-ordinates. Assume first system A.1 undergoes a triple-zero eigenvalue structure

$$\begin{pmatrix} \tilde{r} \\ \tilde{y} \\ \tilde{m} \end{pmatrix} = \mathbf{T} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{A.3}$$

via an appropriate transformation matrix whose columns represent the eigenvectors associated to the triple-zero eigenvalue

$$T = \begin{bmatrix} 1 & 0 & \frac{\gamma}{\beta}\frac{\tau-\beta\gamma}{\tau^2} \\ 0 & \frac{\beta}{\tau} & \frac{\tau-\beta\gamma}{\tau^2} \\ -\beta & \frac{\beta\gamma-\tau}{\tau} & 0 \end{bmatrix} \tag{A.4}$$

which transforms (A.1) into

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \mathbf{F} \tag{A.5}$$

where:

$$\mathbf{F} = \begin{pmatrix} \overline{F}_1\left(w_1, w_2, w_{3,}\right) \\ \overline{F}_2\left(w_1, w_2, w_{3,}\right) \\ \overline{F}_3\left(w_1, w_2, w_{3,}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta\tau}\left(-\tau\gamma + \beta\gamma^2\right)\overline{f}_2\left(w_1, w_2, w_{3,}\right) \\ \gamma\overline{f}_2\left(w_1, w_2, w_{3,}\right) \\ \tau\overline{f}_2\left(w_1, w_2, w_{3,}\right) \end{pmatrix} \tag{A.6}$$

being

$$f_2 = \frac{\alpha}{2}\left[f_{rr}(w_1 + \frac{\gamma}{\beta}\frac{\tau-\beta\gamma}{\tau^2}w_3)^2 + 2f_{ry}(w_1 + \frac{\gamma}{\beta}\frac{\tau-\beta\gamma}{\tau^2}w_3)(\frac{\beta}{\tau}w_2 + \frac{\tau-\beta\gamma}{\tau^2}w_3) + f_{yy}(\frac{\beta}{\tau}w_2 + \frac{\tau-\beta\gamma}{\tau^2}w_3)^2 \right]$$

2. Translation to the origin

Let us define the following parameters translation, $\delta = \overline{\delta} + v$, $\alpha = \overline{\alpha} + \mu$, $g = \overline{g} + \kappa$, such that

$$\begin{pmatrix} \dot{\tilde{r}} \\ \dot{\tilde{y}} \\ \dot{\tilde{m}} \end{pmatrix} = \mathrm{J}^* \begin{pmatrix} \tilde{r} \\ \tilde{y} \\ \tilde{m} \end{pmatrix} + A \begin{pmatrix} \tilde{r} \\ \tilde{y} \\ \tilde{m} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{2}\alpha\left[f_{rr}\tilde{r}^2 + 2f_{ry}\tilde{r}\tilde{y} + f_{yy}\tilde{y}^2 \right] + \frac{1}{2}\mu\kappa \\ 0 \end{pmatrix} \tag{A.7}$$

where

$$A = \begin{bmatrix} -\beta v & \gamma v & -v \\ f_r\mu & f_y\mu & f_m\mu \\ 0 & 0 & 0 \end{bmatrix} \tag{A.8}$$

We can thus construct a versal deformation of the linear part of (A.7) which becomes

$$\mathbf{V}(\tau,\mu,\eta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + M(\tau,\mu,\eta) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{pmatrix} \tag{A.9}$$

Moreover, following Gamero et al. (1999), (A.9) can be normalized to

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= w_3 \\ \dot{w}_3 &= \varepsilon_1 w_1 + \varepsilon_2 w_2 + \varepsilon_3 w_3 + O(w_i^3) \end{aligned} \tag{A.10}$$

which finally describes the unfolding of system (10).